Computing with integrals in nonlinear algebra

#3

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Plan

#1 Differential equations as a datastructure

#2 High-precision numerical evaluation. Application in experimental maths.

#3 Formal integration: diagonals, constants terms, residues

$$\mathbb{C}(t, x_1, \dots, x_n) \longrightarrow \mathbb{C}((t))$$

Iterated Laurent power series

$$\mathbb{C}((t)) = \bigcup_{N\geq 0} t^{-N}\cdot\mathbb{C}[[t]] = \text{Laurent power series field}$$

$$\mathcal{L} \stackrel{\text{def}}{=} \mathbb{C}((x_n))((x_{n-1}))\cdots((x_1)) = \text{Iterated Laurent power series field}$$

(expand first w.r.t. x_1, then x_2, …)

$$\mathbb{C}(x_1,\dots,x_n) \subset \mathcal{L}$$

We can consider rational functions as infinite sums of Laurent monomials.

It depends on the ordering of the variables!

Example $$\frac{1}{x_1+x_2} = \frac{1}{x_1}\cdot\left(\frac{1}{1+\frac{x_2}{x_1}}\right) = \sum_{k=0}^{+\infty}(-1)^k\,\frac{x_2^k}{x_1^{k+1}}$$

Diagonals, constant terms, residues

Let $F = \sum_{\underline{k}\in\mathbb{Z}^n} c_{\underline{k}}\, \underline{x}^{\underline{k}} \in \mathcal{L}$ an iterated Laurent power series

$\operatorname{diag} F = \sum_{k\in\mathbb{Z}} c_{k,\dots,k}\, t^k$ (often considered for $F \in \mathbb{C}[[x_1, \dots, x_n]]$)

$\operatorname{ct}_{x_i} F = \sum_{\underline{k}\in\mathbb{Z}^n} c_{\underline{k}}\, \underline{x}^{\underline{k}}\, [k_i = 0] \in \mathcal{L}$ (without x_i)

$\operatorname{res}_{x_i} F = \sum_{\underline{k}\in\mathbb{Z}^n} c_{\underline{k}}\, \underline{x}^{\underline{k}}\, x_i\, [k_i = -1] \in \mathcal{L}$ (without x_i)

$\operatorname{ct}_{x_i, x_j} F = \operatorname{ct}_{x_i} \operatorname{ct}_{x_j} F$

$\operatorname{res}_{x_i, x_j} F = \operatorname{res}_{x_i} \operatorname{res}_{x_j} F$

(and similarly for more variables)

All related notions!

$$\operatorname{diag} F = \operatorname{ct}_{x_2,\dots,x_n} F\left(\frac{x_1}{x_2 \cdots x_n}, x_2, \dots, x_n\right)$$

$$\operatorname{ct}_{x_i} F = \operatorname{res}_{x_i} \frac{1}{x_i} F$$

Multiple binomial sums

Let $\mathcal{S}$ = functions $\mathbb{Z}^{\mathbb{N}} \longrightarrow \mathbb{Q}$ (sequences with infinitely many indices)

$\mathcal{S}$ contains all sequences with finitely many indices:

if $u : \mathbb{Z}^d \longrightarrow \mathbb{Q}$, define $\tilde{u}(n_1, n_2, \dots) = u(n_1, \dots, n_d)$. so $\tilde{u} \in \mathcal{S}$

Definition. The algebra $\mathcal{B}$ of binomial sums is the smallest subalgebra of $\mathcal{S}$ st.

ⓐ $n \mapsto [n=0] \in \mathcal{B}$ ⓑ $\forall C \in \mathbb{Q}^*$, $n \mapsto C^n \in \mathcal{B}$ ⓒ $(n_1, n_2) \mapsto \binom{n_1}{n_2} \in \mathcal{B}$

ⓓ $\forall \lambda : \mathbb{Z}^d \to \mathbb{Z}^e$ affine $\forall u \in \mathcal{B}$, $\underline{n} \in \mathbb{Z}^d \mapsto u_{\lambda(\underline{n}), 0, \dots} \in \mathcal{B}$

ⓔ $\forall u \in \mathcal{B}$ $\forall d \geq 0$ $(\underline{n}, m) \in \mathbb{Z}^d \times \mathbb{Z} \longrightarrow \sum_{k=0}^{m} u_{\underline{n}, k, 0, \dots} \in \mathcal{B}$

A binomial sum

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \xrightarrow[m\mapsto n]{d} \sum_{k=0}^{m}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3 \xrightarrow{e} \binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3$$

$$\binom{n}{k}$$

$$\binom{n+k}{k} \quad \xleftarrow[n\mapsto n+k]{d} \quad \binom{n}{k}$$

$$\sum_{j=0}^{k}\binom{k}{j}^3 \quad \xleftarrow[m\mapsto k]{d} \quad \sum_{j=0}^{m}\binom{k}{j}^3 \quad \xleftarrow{e} \quad \binom{k}{j}^3$$

$$\binom{k}{j}^3:\quad \binom{k}{j} \qquad \binom{k}{j} \qquad \binom{k}{j}$$

Binomial sums and constant terms

Lemma. Every binomial sum is a linear combination of sequences in the form $n \in \mathbb{Z}^d \longmapsto \operatorname{ct}_{x_1,\dots,x_s}\left(R_0 R_1^{n_1} \cdots R_d^{n_d}\right)$ for some $R_0, \dots, R_d \in \mathbb{C}(x_1,\dots,x_s)$.

Proof. ⓐ $[n=0] = \operatorname{ct}_x x^n$ ⓑ $C^n = \operatorname{ct}_\varnothing C^n$ ⓒ $\binom{n}{k} = \operatorname{ct}_x \left(\frac{1}{x}\right)^k (1+x)^n$

ⓓ $$\operatorname{ct}_{x_1,\dots,x_s} R_0 \prod_i R_i^{\sum_j a_{ij} n_j + b_i} = \operatorname{ct}_{x_1,\dots,x_s} \left(R_0 \prod_i R_i^{b_i}\right) \prod_j \left(\prod_i R_i^{a_{ij}}\right)^{n_j}$$

ⓔ $$\sum_{k=0}^{m} \operatorname{ct}_{x_1\dots x_s} R_0 R_1^{n_1} \cdots R_d^{n_d} T^k = \operatorname{ct}_{x_1\dots x_s} \frac{R_0}{1-T} R_1^{n_1} \cdots R_d^{n_d} - \operatorname{ct}_{x_1\dots x_s} \frac{R_0 T}{1-T} R_1^{n_1} \cdots R_d^{n_d} T^m$$

(Exercise: treat the case $T=1$)

The generating function of a binomial sum

Theorem (Bostan, Lairez, Salvy)

Let $(u_n)_{n\geq 0}$ be a sequence and $f(t) = \sum_{n\geq 0} u_n t^n$ its generating function.

The following are equivalent:

1. u is a binomial sum

$(t \ll x_1 \ll \cdots \ll x_s)$

2. $f(t) = \operatorname{res}_{x_1,\dots,x_s} R(t, x_1, \dots, x_s)$ for some $R \in \mathbb{C}(t, x_1, \dots, x_n)$

3. $f(t) = \operatorname{diag} S(x_1, \dots, x_s)$ for some $S \in \mathbb{C}(x_1,\dots,x_s) \cap \mathbb{C}[[x_1,\dots,x_s]]$

$$\left(\text{For } 1 \Rightarrow 2, \quad \text{use} \quad \sum_{n\geq 0} \operatorname{ct}_{x_1 - x_s}\left(R_0 R_1^n\right) t^n = \operatorname{ct}_{x_1 - x_s} \frac{R_0}{1 - tR_1} \right)$$

Automatic simplification of residues

Let ℓ and $a \in \mathbb{C}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$

If $\ell \ll x_i$, $\quad \dfrac{a}{x_i - \ell} = \dfrac{a}{x_i} \dfrac{1}{1 - \frac{\ell}{x_i}} = \dfrac{a}{x_i} \sum_{n \geq 0} \left(\dfrac{\ell}{x_i}\right)^n \quad \leadsto \operatorname{res}_{x_i} \dfrac{a}{x_i - \ell} = a$

If $\ell \gg x_i$ $\quad \dfrac{a}{x_i - \ell} = -\dfrac{a}{\ell} \dfrac{1}{1 - \frac{x_i}{\ell}} = \dfrac{a}{\ell} \sum_{n \geq 0} \dfrac{x_i^n}{\ell^n} \quad \leadsto \operatorname{res}_{x_i} \dfrac{a}{x_i - \ell} = 0$

Similar rules for higher degree denominators.

(But in some cases no rule applies)

Maple

Residues are differentially finite

Theorem (Christol '85, Lipshitz '88)

For any $F \in \mathbb{C}(x_1, \dots, x_n)$, $\operatorname{diag} F$, $\operatorname{res}_{x_2,\dots,x_n} F$ and $\operatorname{ct}_{x_2,\dots,x_n} F$

are differentially finite.

(And this is effective.)

one case implies the other cases

Corollary · Binomial sums $(u_n)_{n\in\mathbb{N}}$ satisfy linear recurrence relations with poly. coeffs.

· Equality between binomial sums is decidable.

A computational handle on residues

For any $A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$,

$$\operatorname*{res}_{x_1,\dots,x_n} \left(\frac{\partial A_1}{\partial x_1} + \dots + \frac{\partial A_n}{\partial x_n} \right) = 0$$

Note also that for any $L(t, \partial_t) \in \mathcal{D}$ (ring of differential operators),

$L \cdot \operatorname*{res}_x F = \operatorname*{res}_x (L \cdot F)$, for any $F \in \mathbb{C}(t, x_1, \dots, x_n)$

Corollary. For any $F \in \mathbb{C}(t, x_1, \dots, x_n)$ and $L(t, \partial_t) \in \mathcal{D}$,

$\exists\, A_1, \dots, A_n \in \mathbb{C}(t, x_1, \dots, x_n)$ s.t. $L \cdot F = \sum_i \frac{\partial A}{\partial x_i} \Rightarrow L \cdot \operatorname*{res}_x F = 0$

the source of all known algorithms to compute residues

Proof of the theorem

Lipshitz's proof

$F \doteq \frac{A}{P} \in \mathbb{C}(t, x_1, \dots, x_n)$. $\quad \delta \doteq \max(\deg_x A, \deg_x P)$

1 $\quad V_N \doteq \frac{\mathbb{C}(t)[x_1, \dots, x_n]_{\leq N\delta}}{P^N} \subset \mathbb{C}(t, x_1, \dots, x_n) \qquad \dim_{\mathbb{C}(t)} V_N = \binom{N\delta + n}{n} \underset{N\to\infty}{\sim} \frac{1}{n!}(N\delta)^n$

$D_N \doteq \left\{ \frac{\partial^{\alpha+\beta_1+\dots+\beta_n} F}{\partial t^\alpha \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \;\middle|\; \alpha + \beta_1 + \dots + \beta_n \leq N-1 \right\} \qquad \# D_N = \binom{N+n}{n+1} \underset{N\to\infty}{\sim} \frac{1}{(n+1)!} N^{n+1}$

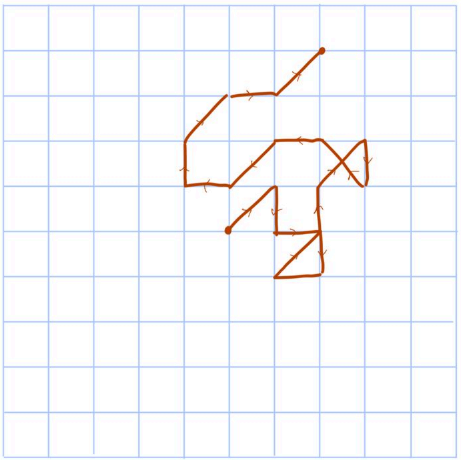
2. As $N \to \infty$, $\quad \dim V_N = o(\# D_N)$

so $\exists\, c_{\alpha,\underline{\beta}}(t) \in \mathbb{C}(t)$ s.t. $\displaystyle\sum_{\alpha,\underline{\beta}} c_{\alpha,\beta}(t) \frac{\partial^{\alpha+|\beta|} F}{\partial t^\alpha \partial \underline{x}^{\underline{\beta}}} = 0$ $\;*\;$ (some $c_{\alpha,\beta} \neq 0$)

3. Let $L = \sum_\alpha c_{\alpha,\underline{0}}(t) \partial_t^\alpha$. $\qquad L \cdot F = \displaystyle\sum_{\substack{\alpha,\underline{\beta} \\ \underline{\beta} \neq 0}} c_{\alpha,\beta}(t) \frac{\partial^{\alpha+|\beta|} F}{\partial t^\alpha \partial \underline{x}^{\underline{\beta}}} = \frac{\partial A_1}{\partial x_1} + \dots + \frac{\partial A_n}{\partial x_n}$

⚠ If $L = 0$, multiply $*$ by $\underline{x}^{\underline{\gamma}}$ for some minimal γ s.t. $c_{\alpha,\underline{\gamma}} \neq 0$

Application in combinatorics: king walk (unconstrained)



$$\sum_n Q_n(x,y)\, t^n$$

$$\parallel$$

$$Q(t,x,y) \doteq \sum_{n,i,j} a_{n,i,j}\, t^n x^i y^j \in \mathbb{Q}(x,y)[[t]]$$

$$S(x,y) \doteq x + yx + y + \frac{y}{x} + \frac{1}{x} + \frac{1}{xy} + \frac{x}{y}$$

Lemma. $Q_{n+1} = S\, Q_n.$ $\quad (Q = tSQ + 1)$

$$\rightsquigarrow \quad Q = \frac{1}{1-tS}$$

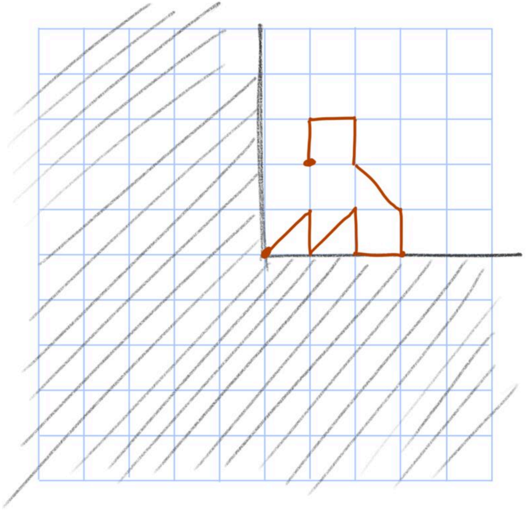
a_n = # of paths of length n starting from $(0,0)$

$a_{n,i,j}$ = # of paths of length n from $(0,0)$ to (i,j)
ending at (i,j)

$$\sum_n a_n t^n = Q(t,1,1) = \frac{1}{1-8t}$$

$$a_n = 8^n$$

King walk in the quarter plane



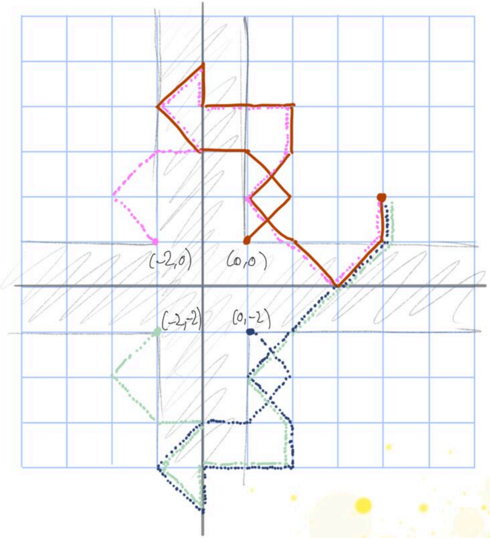
b_n = # of path of length n starting from $(0,0)$
and staying in the quarter plane.

$b_{n,i,j}$ = # of such paths ending at (i,j)

$$Q(t,x,y) \doteq \sum_{n,i,j} b_{n,i,j}\, t^n x^i y^j$$

What is the nature of $Q(t,1,1)$?

King walk in the quarter plane



Reflection principle. For any $i,j \geqslant 0$,

$$\underbrace{b_{n,i,j} = \#\{(0,0) \xrightarrow{n} (i,j)\} - \#\{(-2,0) \xrightarrow{n} (i,j)\} - \#\{(0,-2) \xrightarrow{n} (i,j)\} + \#\{(-2,-2) \xrightarrow{n} (i,j)\}}_{= c_{n,i,j} \quad (\text{defined for } i,j \in \mathbb{Z})}$$

$$C(t,x,y) = \sum_{n \geqslant 0} \sum_{i,j \in \mathbb{Z}} c_{n,i,j}\, t^n x^i y^j = \frac{1 - x^{-2} - y^{-2} + x^{-2}y^{-2}}{1 - tS}$$

Lemma. $Q(t,1,1) = \operatorname{res}_{x,y} \dfrac{C(t,\frac{1}{x},\frac{1}{y})}{x(1-x)\,y(1-y)}$

$\rightsquigarrow$ $Q(t,1,1)$ is differentially finite.

Residues are integrals

At last an integral!

For any $F \in \mathbb{C}(t, x_1, \dots, x_n)$,

$$\operatorname*{res}_{x_1,\dots,x_n} F = \frac{1}{(2i\pi)^n} \oint_{|x_1|=r_1} \cdots \oint_{|x_n|=r_n} F(t, x_1, \dots, x_n)\, dx_n \cdots dx_1$$

with $|t| \ll r_1 \ll \dots \ll r_n \ll 1$

Corollary For any $F \in \mathbb{C}(t,x)$, $\operatorname*{res}_x F$ is algebraic

Proof. $F = \dfrac{A(t,x)}{P(t,x)^N}$ $\quad P = a(t) \prod_{i=1}^{d} (x - \rho_i(t))$, where $a(t) \in \mathbb{C}(t)$ and $\rho_1, \dots, \rho_d$ are algebraic

$$\operatorname*{res}_x F = \frac{1}{2i\pi} \oint_{|x|=r} F(t,x) = \sum_{\substack{i=1 \\ \text{s.t. } |\rho_i(t)| < r}}^{d} \operatorname*{Res}_{x=\rho_i} F \in \mathbb{C}(t)(\rho_1, \dots, \rho_d).$$

Arithmetic properties of residues

Regularity theorem (Griffiths). Let $F \in \mathbb{C}(t, x_1, \ldots, x_n)$

The minimal operator annihilating $\operatorname{res}_{x_1 \ldots x_n} F$

is Fuchsian with rational exponents.

Algebraicity theorem (Furstenberg) Let $F \in \mathbb{Q}(t, x_1, \ldots, x_n)$

For all but finitely many prime p, the reduction

in $\mathbb{F}_p((t))$ of $\operatorname{res}_{x_1 - x_n} F$ exists. When it does,

it is algebraic.

Extraction operators (towards Furstenberg's theorem)

Let $F \in \mathbb{F}_p(t, x_1, \dots, x_n)$ and $f = \underset{x_1,\dots,x_n}{\mathrm{ct}} F$

$$E_r\Big(\sum_k a_k t^k\Big) \overset{\text{def}}{=} \sum_k a_{pk+r} t^k \quad ; \quad E_r\Big(\sum_{k,\underline{m}} c_{k,\underline{m}} t^k \underline{x}^{\underline{m}}\Big) \overset{\text{def}}{=} \sum_{k,\underline{m}} c_{pk+r,p\underline{m}} t^k \underline{x}^{\underline{m}}$$

1. $\underset{x_1,\dots,x_n}{\mathrm{ct}} \circ E_r = E_r \circ \underset{x_1,\dots,x_n}{\mathrm{ct}}$

2. $E_r(A^p G) = A\, E_r(G)$, for any $A, G \in \mathbb{F}_p(t, x_1, \dots, x_n)$

$\big($ NB: $A(t, x_1, \dots, x_n)^p = A(t^p, x_1^p, \dots, x_n^p)\big)$

3. $f(t) = \sum_{r=0}^{p-1} t^r E_r(f)^p$, for any $f \in \mathbb{F}_p((t))$

Proof of Furstenberg's theorem

Let $F = \dfrac{A}{P} \in \mathbb{F}_p(t, x_1, \dots, x_n)$ $\qquad \delta = \max\left(\deg_{t,x} A, \deg_{t,x} P\right)$ $\qquad f = \underset{x_1,\dots,x_n}{\mathrm{ct}} F$

Let $\mathcal{E} = \left\{ \underset{x_1,\dots,x_n}{\mathrm{ct}} \dfrac{B}{P} \;\middle|\; B \in \mathbb{F}_p[t, x_1, \dots, x_n]_{\leq \delta} \right\}$

1. $\forall r, \quad E_r(\mathcal{E}) \subseteq \mathcal{E}$ $\qquad E_r\left(\mathrm{ct}\, \dfrac{B}{P}\right) = \mathrm{ct}\left(E_r\left(\dfrac{B}{P}\right)\right) = \mathrm{ct}\left(\dfrac{E_r(BP^{p-1})}{P}\right)$

Let $g_1, \dots, g_s \in \mathcal{E}$ be a basis of $\mathcal{E}$

2. There are $c_{i,j} \in \mathbb{F}_p[t]$ s.t. $g_i = \sum_{j=1}^{s} c_{ij}\, g_j^p$

$$g_i = \sum_{r=0}^{p-1} t^r E_r(g_i)^p = \sum_{r=0}^{p-1} t^r \left(\sum_j b_{ij} g_j\right)^p$$

$$= \sum_j \left(\sum_r b_j^p t^r\right) g_j^p$$

3. All the elements of $\mathcal{E}$ are algebraic.

Over $\mathbb{F}_p(t)$, $\quad \mathrm{Vect}\left\{ g_i^{p^k} \mid 1 \leq i \leq s,\ k < N \right\} \subseteq \mathrm{Vect}\left\{ g_i^{p^N} \mid 1 \leq i \leq s \right\}$

Algorithm: Hermite's reduction

Given $F \in \mathbb{C}(t,x)$ how to compute $\operatorname{res}_x F$ efficiently?

Write $F = \frac{A}{P^N}$, with $A, P \in \mathbb{C}[t,x]$ and P square-free

NB: The base field could be anything (char. 0), $\mathbb{C}(t)$ is not important here, we just need to recognize derivatives.

Reduction rules (modulo derivatives)

$$x^k = \frac{\partial}{\partial x}\left(\frac{1}{k+1} x^{k+1}\right) \longrightarrow 0 \qquad \frac{B}{P} = \frac{QP+R}{P} \longrightarrow \frac{R}{P}$$

$$\frac{A}{P^N} = \frac{UP + V\frac{\partial}{\partial x}P}{P^N} = \frac{U}{P^{N-1}} + \frac{1}{N-1}\frac{\frac{\partial}{\partial x}V}{P^{N-1}} - \frac{\partial}{\partial x}\left(\frac{1}{N-1}\cdot\frac{V}{P^{N-1}}\right) \longrightarrow \frac{U + \frac{1}{N-1}\frac{\partial}{\partial x}V}{P^{N-1}}$$

Algorithm. $F \xrightarrow{*} \frac{R_0}{P}$, $\frac{\partial}{\partial t}F \xrightarrow{*} \frac{R_1}{P}$, ... $\rightsquigarrow$ Finite dimensional confinement

$\rightsquigarrow$ $\exists c_0(t), \dots, c_s(t) \in \mathbb{C}(t)$ s.t. $\sum_i c_i(t)\frac{\partial^i F}{\partial t^i} = \frac{\partial}{\partial}(\dots)$

An insight into higher dimensional residues

Let $F = \frac{A}{P^N} \in K[x_1, \dots, x_n]$ $\quad (K = \mathbb{Q}, \mathbb{C}(t), \dots)$

How to decide if $F = \frac{\partial}{\partial x_1} \frac{B_1}{P^M} + \dots + \frac{\partial}{\partial x_n} \frac{B_n}{P^M}$ for some $B_i \in K[x_1, \dots, x_n]$ and $M > 0$?

First idea: Try to write $A = BP + \sum_i C_i \frac{\partial}{\partial x_i} P$

so that $\frac{A}{P^N} = \frac{B + \frac{1}{N-1} \sum_i \frac{\partial}{\partial x_i} C_i}{P^{N-1}} + \sum_i \frac{\partial}{\partial x_i} (\dots)$

Griffiths-Dwork reduction.

Second idea: If you have a nontrivial relation $AP^r = BP + \sum_i C_i \frac{\partial P}{\partial x_i}$

then $\frac{A}{P^N} = \frac{B + \frac{1}{N+r-1} \sum_i \frac{\partial}{\partial x_i} C_i}{P^{N+r-1}} + \sum_i \frac{\partial}{\partial x_i} (\dots)$