Diagonals of rational functions

Main Conference of Chaire J. Morlet
Artin approximation and infinite dimensional geometry
27 mars 2015

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Diagonals: definitions and properties

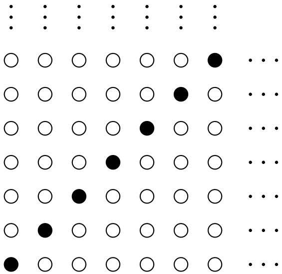
Binomial sums

Computing diagonals

Diagonal of a power series

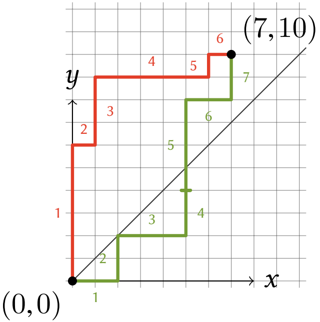
Définition

- $f = \sum_{i_1,\dots,i_n \in \mathbb{N}^n} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Q}[[x_1,\dots,x_n]]$
- $\operatorname{diag} f \stackrel{\text{def}}{=} \sum_{i \geqslant 0} a_{i,\dots,i} t^i$



A combinatorial problem

Counting rook paths



$a_{i,j} \stackrel{\text{def}}{=}$ nb. of rook paths from $(0,0)$ to (i,j)

Easy recurrence:

$$a_{i,j} = \sum_{k<i} a_{k,j} + \sum_{k<j} a_{i,k}$$

What about $a_{n,n}$? asymptotic? existence of a recurrence?

Recurrence relations for rook paths

- dimension 2

$$9nu_n - (14+10n)u_{n+1} + (2+n)u_{n+2} = 0$$

- dimension 3

$$\begin{aligned} &-192n^2(1+n)(88+35n)u_n \\ &+(1+n)(54864+100586n+59889n^2+11305n^3)u_{n+1} \\ &-(2+n)(43362+63493n+30114n^2+4655n^3)u_{n+2} \\ &+2(2+n)(3+n)^2(53+35n)u_{n+3} = 0 \end{aligned}$$

- dimension 4

$$\begin{aligned} &5000n^3(1+n)^2(2705080+3705334n+1884813n^2+421590n^3+34983n^4)u_n \\ &-(1+n)^2(80002536960+282970075928n+\cdots+6386508141n^6+393838614n^7)u_{n+1} \\ &+2(2+n)(143370725280+500351938492n+\cdots+2636030943n^7+131501097n^8)u_{n+2} \\ &-(3+n)^2(26836974336+80191745800n+100381179794n^2+\cdots+44148546n^7)u_{n+3} \\ &+2(3+n)^2(4+n)^3(497952+1060546n+829941n^2+281658n^3+34983n^4)u_{n+4} = 0 \end{aligned}$$

Differential equation for diagonals

$$a_{i,j} = \sum_{k<i} a_{k,j} + \sum_{k<j} a_{i,k} \Rightarrow \sum_{i,j\geqslant 0} a_{i,j} x^i y^j = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}$$

$$\sum_{n\geqslant 0} a_{n,n} t^n = \operatorname{diag}\left(\frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}\right).$$

Theorem (Lipshitz 1988) — "diagonal ⇒ differentially finite"

If $R \in \mathbb{Q}(x_1, \dots, x_n) \cap \mathbb{Q}[[x_1, \dots, x_n]]$, then $\operatorname{diag} R$ satisfies a linear differential equation with polynomial coefficients
$c_r(t) y^{(r)} + \cdots + c_1(t) y' + c_0(t) y = 0.$

More properties of diagonals

Theorem (Furstenberg 1967) — "algebraic ⇒ diagonal"

If $f(t) = \sum a_n t^n$ is an algebraic series (i.e. $\mathrm{P}(t, f(t)) = 0$ for some $\mathrm{P} \in \mathbb{Q}[x,y]$), then it is the diagonal of a rational power series.

Theorem (Furstenberg 1967) — "diagonal ⇒ algebraic mod p"

If $\sum a_n t^n \in \mathbb{Q}[\![t]\!]$ is the diagonal of a rational power series, then it is an algebraic series modulo p for almost all prime p.

Algebricity modulo p

Example

$$f = \sum_n \frac{(3n)!}{n!^3} t^n = \operatorname{diag}\left(\frac{1}{1-x-y-z}\right) \text{ is } \textit{not} \text{ algebraic.}$$

- $f \equiv (1+t)^{-\frac{1}{4}} \mod 5$
- $f \equiv \left(1+6t+6t^2\right)^{-\frac{1}{6}} \mod 7$
- $f \equiv \left(1+6t+2t^2+8t^3\right)^{-\frac{1}{10}} \mod 11$
- ...

Besides, $\left(27t^2 - t\right) f'' + (54t-1) f' + 6f = 0.$

Proof of algebricity modulo p

$\mathbb{F}_q$, the base field

Slicing operators — For $r \in \mathbb{Z}$,

$$\mathrm{E}_r\left(\sum_i a_i t^i\right) \overset{\text{def}}{=} \sum_i a_{qi+r} t^i \text{ and } \mathrm{E}_r\left(\sum_{\mathrm{I}} a_{\mathrm{I}} \underline{x}^{\mathrm{I}}\right) \overset{\text{def}}{=} \sum_{\mathrm{I}} a_{q\mathrm{I}+(r,\dots,r)} \underline{x}^{\mathrm{I}}$$

We check

- $\operatorname{diag} \circ \mathrm{E}_r = \mathrm{E}_r \circ \operatorname{diag}$;
- $x_i \mathrm{E}_r(\mathrm{F}) = \mathrm{E}_r(x_i^q \mathrm{F})$;
- $\mathrm{G}(\underline{x})\mathrm{E}_r(\mathrm{F}) = \mathrm{E}_r(\mathrm{G}(\underline{x})^q \mathrm{F})$, because $\mathrm{G}(\underline{x}^q) = \mathrm{G}(\underline{x})^q$, where $\underline{x}^q = x_1^q, \dots, x_n^q$;
- If $f(t) = \sum_i a_i t^i$, then

$$f(t) = \sum_{0 \leqslant r < q} t^r \sum_i a_{qi+r} t^{qi}$$

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- If $f(t) = \sum_i a_i t^i$, then

$$f(t) = \sum_{0 \leqslant r < q} t^r \left(\sum_i a_{qi+r} t^i\right)^q$$

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- If $f(t) = \sum_i a_i t^i$, then

$$f(t) = \sum_{0 \leqslant r < q} t^r \mathrm{E}_r(f)^q$$

Proof of algebricity modulo p

Let $R = \frac{A}{F} \in \mathbb{F}_q(\underline{x})$, $d = \max(\deg A, \deg F)$ and the $\mathbb{F}_q$-vector space

$$\mathcal{V} = \left\{ \operatorname{diag}\left(\tfrac{P}{F}\right) \,\middle|\, \deg P \leqslant d \right\} \subset \mathbb{F}_q[[t]] \qquad \left(\dim \mathcal{V} \leqslant \binom{d+n}{n}\right)$$

1. Operators E_r stabilize $\mathcal{V}$.

   Proof.

$$\begin{aligned} E_r \circ \operatorname{diag}\left(\frac{P}{F}\right) &= \operatorname{diag} \circ E_r\left(\frac{PF^{q-1}}{F^q}\right) \\ &= \operatorname{diag}\left(\frac{E_r(PF^{q-1})}{F}\right) \in \mathcal{V} \end{aligned}$$

Proof of algebricity modulo p

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1. Operators E_r stabilize $\mathcal{V}$.
2. Let $f_1, \dots, f_s$ be a basis of $\mathcal{V}$. There exist $c_{ij} \in \mathbb{F}_q[t]$ such that

$$\forall i, f_i = \sum_j c_{ij} f_j^q.$$

Proof.

$$f_i = \sum_{0 \leqslant r < q} t^r \mathrm{E}_r(f_i)^q = \sum_{0 \leqslant r < q} t^r \left(\sum_j b_{ij} f_j \right)^q$$

$$= \sum_j \left(\sum_{0 \leqslant r < q} b_{ij} t^r \right) f_j^q$$

Proof of algebricity modulo p

Let $R = \frac{A}{F} \in \mathbb{F}_q(\underline{x})$, $d = \max(\deg A, \deg F)$ and the $\mathbb{F}_q$-vector space

$$\mathcal{V} = \left\{ \operatorname{diag}\left(\tfrac{P}{F}\right) \,\middle|\, \deg P \leqslant d \right\} \subset \mathbb{F}_q[\![t]\!] \qquad \left(\dim \mathcal{V} \leqslant \tbinom{d+n}{n}\right)$$

1. Operators E_r stabilize $\mathcal{V}$.
2. Let $f_1, \ldots, f_s$ be a basis of $\mathcal{V}$. There exist $c_{ij} \in \mathbb{F}_q[t]$ such that

$$\forall i, f_i = \sum_j c_{ij} f_j^q.$$

3. All the elements of $\mathcal{V}$ are algebraic.
   Proof.

$$\forall i, f_i^q = \sum_j c_{ij}^q f_j^{q^2}, \text{ etc.}$$

Thus, over $\mathbb{F}_q(t)$, $\operatorname{Vect}\left\{ \Delta(R)^{q^k} \,\middle|\, 0 \leqslant k \leqslant s \right\} \subset$
$\operatorname{Vect}\left\{ f_i^{q^k} \,\middle|\, 0 \leqslant k \leqslant s,\ 1 \leqslant i \leqslant s \right\} \subset \operatorname{Vect}\left\{ f_i^{q^s} \,\middle|\, 1 \leqslant i \leqslant s \right\}.$

Characterization of diagonals?

Conjecture (Christol 1990)

"integer coefficients + convergent + diff. finite ⇒ diagonal"

If $\sum a_n t^n \in \mathbb{Z}[[t]]$, has radius of convergence > 0, and satisfies a linear differential equation with polynomial coefficients, then it is the diagonal of a rational power series.

A hierarchy of power series — For $f \in \mathbb{Q}[[t]]$, let $\mathrm{N}(f)$ be the minimum number of variables $x_1, \ldots, x_{\mathrm{N}(f)}$ such that $f = \operatorname{diag} \mathrm{R}(x_1, \ldots, x_{\mathrm{N}(f)})$, with R rational power series, if any.

- $\mathrm{N}(f) = 1 \Leftrightarrow f$ is rational
- $\mathrm{N}(f) = 2 \Leftrightarrow f$ is algebraic irrational
- $\mathrm{N}\left(\sum_n \frac{(3n)!}{n!^3} t^n\right) = 3$
- **Question** : Find a f such that $3 < \mathrm{N}(f) < \infty$.

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Examples

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3} \qquad \text{(Dixon)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \qquad \text{(Strehl)}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2$$

$$\sum_{r\geqslant 0}\sum_{s\geqslant 0} (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n} = \sum_{k\geqslant 0} \binom{n}{k}^4$$

Binomial sums

Further examples

Number theory

$$n^3 u_n + (n-1)^3 u_{n-2} = (34n^3 - 51n^2 + 27n - 5) u_{n-1}$$

$$\text{avec } u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \text{(Apéry)}$$

Important step in proving that $\zeta(3) = \sum_{n\geqslant 1} \frac{1}{n^3} \notin \mathbb{Q}$

Analysis of algorithm

> *[50] Develop computer programs for simplifying sums that involve binomial coefficients.*
>
> Exercise 1.2.6.63
> *The Art of Computer Programming*
> Knuth (1968)

Definition

- $\delta : \mathbb{Z} \to \mathbb{Q}$ is a binomial sum. ($\delta_0 = 1$ et $\delta_n = 0$ si $n \neq 0$)
- $n \in \mathbb{Z} \mapsto a^n \in \mathbb{Q}$ is a binomial sum for all $a \in \mathbb{Q}^\times$.
- $(n,k) \in \mathbb{Z}^2 \mapsto \binom{n}{k} \in \mathbb{Q}$ is a binomial sum.

- If $u, v : \mathbb{Z}^p \to \mathbb{Q}$ are b.s., then $u + v$ and uv are b.s.
- If $u : \mathbb{Z}^p \to \mathbb{Q}$ are b.s. and $\lambda : \mathbb{Z}^p \to \mathbb{Z}^q$ is an affine map, then $u \circ \lambda$ is a b.s.
- If $u : \mathbb{Z}^p \to \mathbb{Q}$ is a b.s., then

$$(n_1, \ldots, n_p) \in \mathbb{Z}^p \mapsto \sum_{k=0}^{n_1} u_{k,n_2,\ldots,n_p} \in \mathbb{Q}$$

is a b.s.

Diagonals ↔ binomial sums

Theorem (Bosta, Lairez, Salvy 2014) — A sequence $(u_n)_{n\geqslant 0}$ is a binomial sum **if and only if** its generating function $\sum u_n t^n$ is the diagonal of a rational series.

Example

$$\sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\right)t^n = \operatorname{diag}\left(\frac{1}{(1-x_1)(1-x_2)(1-x_3)-x_4(x_1+x_2x_3-x_1x_2x_3)}\right)$$

Corollaries

Corollary 1 — “binomial sum ⇒ recurrence”

If $(u_n)_{n\geqslant 0}$ is a binomial sum, then it satisfies a linear recurrence with polynomial coefficients.

Corollary 2 — “algebraic g.f. ⇒ binomial sum”

If $\sum u_n t^n$ is an algebraic series then $(u_n)_{n\geqslant 0}$ is a binomial sum.

Corollary 3 — “binomial sum ⇒ algebraic g.f. mod p”

If $(u_n)_{n\geqslant 0}$ is a binomial sum, then $\sum u_n t^n$ is an algebraic series modulo p for almost all prime p.

Conjecture

“integral + exp. bounded + recurrence ⇒ binomial sum”

If $(u_n)_{n\geqslant 0} \in \mathbb{Z}^{\mathbb{N}}$ grows at most exponentially and satisfies a linear recurrence with polynomial coefficients, then it is a binomial sum.

Binomial sums as diagonals

Skectch of the proof

Proposition — All binomial sums are linear combinations of sequences in the form $(k_1, \ldots, k_p) \in \mathbb{Z}^p \mapsto [1]\left(R_0 R_1^{k_1} \cdots R_d^{k_d}\right)$, where $R_0, \ldots, R_p \in \mathbb{Q}(x_1, \ldots, x_d)$.

With one variable,

$$[1]\left(RS^n\right) = [x_1^n \ldots x_{d+1}^n]\left(\underbrace{\frac{R}{1 - x_1 \cdots x_{d+1} S}}_{\text{may not be a power series}}\right)$$

Diagonals as binomial sums

Skectch of the proof

- Let $R = \dfrac{\underline{x}^{\underline{m}_0}}{1 + a_1\underline{x}^{\underline{m}_1} + \cdots + \underline{x}^{\underline{m}_r}} \in \mathbb{Q}(x_1, \ldots, x_d)$
- $R = \underline{x}^{\underline{m}_0} \sum_{\underline{k}\in\mathbb{N}^r} \underbrace{\binom{k_1+\cdots+k_r}{k_1,\ldots,k_r} a_1^{k_1} \cdots a_r^{k_r}}_{C_{\underline{k}}} \underline{x}^{k_1\underline{m}_1} \cdots \underline{x}^{k_r\underline{m}_r}$
- $\binom{k_1+\cdots+k_r}{k_1,\ldots,k_r} = \binom{k_1+\cdots+k_r}{k_1}\binom{k_2+\cdots+k_r}{k_2}\cdots\binom{k_{r-1}+k_r}{k_r}$, so $C_{\underline{k}}$ is a binomial sum
- $[x_1^n \cdots x_d^n]R = \sum_{k\in\mathbb{Z}^e} C_{\underline{k}} \mathbb{1}_\Gamma(n,\underline{k})$ où

$$\Gamma = \left\{(n,\underline{k}) \in \mathbb{R}\times\mathbb{R}_+^e \;\middle|\; \underline{m}_0 + \sum_{i=1}^{e} k_i\underline{m}_i = (n,\ldots,n)\right\}.$$

Diagonals: definitions and properties

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Algorithmic Lipshitz' theorem

Theorem (Lipshitz 1988) — "diagonal ⇒ differentially finite"

If $R \in \mathbb{Q}(x_1, \ldots, x_n) \cap \mathbb{Q}[[x_1, \ldots, x_n]]$, then diag R satisfies a linear differential equation with polynomial coefficients

$c_r(t)y^{(r)} + \cdots + c_1(t)y' + c_0(t)y = 0.$

- How to compute the differential equation?
- Once computer, we get *everything* we want about the diagonal.
- It would allow to prove identities between binomial sums.

Diagonals as integrals

- Basic fact: $\forall C,\ \operatorname{diag}\left(x_i \frac{\partial C}{\partial x_i} - x_j \frac{\partial C}{\partial x_j}\right) = 0$
- Consider the following transformation

$$\begin{aligned} \mathcal{T} : \mathbb{Q}(x_1, \ldots, x_n) &\to \mathbb{Q}(t, x_1, \ldots, x_{n-1}) \\ R &\mapsto \frac{1}{x_1 \cdots x_{n-1}} R\left(x_1, \ldots, x_{n-1}, \frac{t}{x_1 \cdots x_{n-1}}\right). \end{aligned}$$

- $\mathcal{T}(R) = \sum_{i=1}^{n-1} \frac{\partial C_i}{\partial x_i} \Rightarrow \operatorname{diag} R = 0.$
- $\operatorname{diag} R = \frac{1}{(2i\pi)^n} \oint \mathcal{T}(R) \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1}$

Computing integrals

- $\mathbb{K}$ a field of characteristic 0 with a derivation δ (usually $\mathbb{K} = \mathbb{Q}(t)$ and $\delta = \frac{\partial}{\partial t}$).
- $R = \frac{a}{f} \in \mathbb{K}(x_1, \ldots, x_n)$

Problem — Find $c_0, \ldots, c_r \in \mathbb{K}$ such that $\exists C_1, \ldots, C_n \in \mathbb{K}(\underline{x})$

$$c_r \delta^r(R) + \cdots + c_1 \delta(R) + c_0(R) = \sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i}.$$

Problem (bis) — Compute a basis and normal forms in

$$\mathbb{K}(x_1, \ldots, x_n) / \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathbb{K}(x_1, \ldots, x_n)$$

Computing integrals

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Problem (bis) — Compute a basis and normal forms in

$$\mathbb{K}[x_1, \ldots, x_n, f^{-1}] / \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathbb{K}[x_1, \ldots, x_n, f^{-1}]$$
$$=: H^n_{\mathrm{Rham}}(\mathbb{A}^n_{\mathbb{K}} \setminus V(f)) \simeq H^{n-1}_{\mathrm{Rham}}(V(f))$$

Finiteness of the de Rham cohomology

Theorem (Grothendieck 1966) — $\mathrm{H}^n_{\mathrm{Rham}}(\mathbb{A}^n_{\mathbb{K}} \setminus \mathrm{V}(f))$ is a finite dimensional $\mathbb{K}$-vector space.

Theorem (Griffiths 1969)

$$\dim_{\mathbb{K}} \mathrm{H}^n_{\mathrm{Rham}}(\mathbb{A}^n_{\mathbb{K}} \setminus \mathrm{V}(f)) < (\deg f + 1)^n$$

Corollary — The diagonal of $\mathrm{R}(x_0, \ldots, x_n)$ is solution of a linear differential equation with polynomial coefficients of order at most $(d+1)^n$, where d is the degree of the denominator of $\mathcal{T}(\mathrm{R})$.

Reduction of pole order

Homogeneous case

- $f \in \mathbb{K}[x_0, \ldots, x_n]$ homogeneous
- $V_f \stackrel{\text{def}}{=} \left\{ \frac{a}{f^q} \text{ homogeneous of degree } -n-1 \right\}$
- Notation : $\partial_i \stackrel{\text{def}}{=} \frac{\partial}{\partial x_i}$
- Fact : $\partial_i \dfrac{c}{f^{q-1}} = \dfrac{\partial_i c}{f^{q-1}} - (q-1)\dfrac{c \partial_i f}{f^q}$

Rewriting rule — $\dfrac{\sum_i c_i \partial_i f}{\boxed{f^q}} \to \dfrac{1}{q-1} \dfrac{\sum_i \partial_i c_i}{\boxed{f^{q-1}}}$ (maps $V_f \to V_f$)

Theorem (Griffiths 1969) — If $V(f) \subset \mathbb{P}^n_{\mathbb{K}}$ is smooth then

$$\forall \frac{a}{f^q} \in V_f, \quad \frac{a}{f^q} = \sum_i \partial_i \frac{c_i}{f^s} \Rightarrow \frac{a}{f^q} \to^* 0.$$

Reduction of pole order

Homogeneous singular case

- Rewriting rules are ambiguous: if $\sum_i c_i \partial_i f = 0$, then $0 \to \frac{\sum_i \partial_i c_i}{f^q}$.
- We can add the rules $\frac{\sum_i \partial_i c_i}{f^q} \xrightarrow{\text{rg } 2} 0$, it still preserves equivalence classes modulo derivatives.
- New reductions $0 \xrightarrow{\text{rk } r} \frac{b}{f^q}$ appear, we add the rules $\frac{b}{f^q} \xrightarrow{\text{rk } r+1} 0$.

Theorem — There exists an $r > 0$ such that for all $\frac{a}{f^q} \in \mathrm{V}_f$

$$\forall \frac{a}{f^q} \in \mathrm{V}_f, \quad \frac{a}{f^q} = \sum_i \partial_i \frac{c_i}{f^s} \Rightarrow \frac{a}{f^q} \xrightarrow{\text{rk } r}{}^{*} 0.$$

$\rightsquigarrow$ Leads to an efficient algorithm for computing rational integrals (Lairez 2015).

An example

$$f = 2xyz(w-x)(w-y)(w-z) - w^3(w^3 - w^2z + xyz)$$

$e(q,r)$: number of indepedent rational functions a/f^q that are not reducible with rules of rank r

q	0	1	2	3	4	$q > 4$
no rule	0	10	165	680	1771	$\sim 36q^3$
$e(q,1)$	0	10	86	102	120	$\sim 18q$
$e(q,2)$	0	10	7	6	6	6
$e(q,3)$	0	9	1	0	0	0

- $\dim \mathrm{H}^3(\mathbb{P}^3 \setminus \mathrm{V}(f)) = 10$