A polynomial time algorithm for rational creative telescoping

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creative telescoping

General framework to handle multiple integrals with parameters in computer algebra.

rational

We restrict ourselves to rational integrands.

polynomial time algorithm

Polynomial with respect to the generic size of the output.

Multiple rational integrals

Problem

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \ldots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ F(t, \mathbf{x}) & \text{— rational function} \\ \gamma & \text{— a } n\text{-cycle in } \mathbb{C}^n \end{array}\right\} \oint_\gamma F(t, \mathbf{x}) \mathrm{d}\mathbf{x}$$

How to compute this integral?

Theorem (Picard)

These integrals satisfy linear differential equations with polynomial coefficients.

The “why”

Rational–algebraic equivalence

n-integrals of algebraic functions are $(n+1)$-tuple integrals of rational functions.

Combinatorics Differential approach to discrete identities like

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 .$$

(Strehl)

Physics Computation of various special functions, like “n-particle contribution to the magnetic susceptibility of the Ising model”.

Number theory Computation of mirror maps.

Algebraic geometry Computation of topological invariants.

Examples

Univariate integrals

$\oint F(t,x)\mathrm{d}x$ is an algebraic function of t (by residue theorem).

Perimeter of an ellipse

Perimeter of an ellipse with excentricity e and semi-major axis 1:

$$p(e) = \int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\mathrm{d}x \propto \oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}},$$

$$(e-e^3)p'' + (1-e^2)p' + ep = 0 \qquad \text{(Euler, 1733)}$$

The "how"

How to compute algebraically an analytical object?

Fact

For all rational functions $A(t, \mathbf{x})$ finite on γ,

$$\oint_\gamma \frac{\partial A}{\partial x_i} \mathrm{d}\mathbf{x} = 0.$$

The “how”

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \dots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ F(t, \mathbf{x}) & \text{— rationnal function} \\ \gamma & \text{— a } n\text{-cycle} \end{array}\right\} \oint_\gamma F(t, \mathbf{x}) \mathrm{d}\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k F}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \quad \Rightarrow \quad \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma F \mathrm{d}\mathbf{x} = 0$$

We want to:

1. find the $c_k(t)$ which satisfy the telescopic relation,
2. without computing the certificate (A_i).

Example

Perimeter of an ellipse

$$p(e) \propto \oint \frac{\mathrm{d}y\mathrm{d}x}{1 - \frac{1-e^2x^2}{(1-x^2)y^2}}$$

Telescopic relation:

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right) \cdot \left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x \left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\, \partial_y \left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0$.

Brief review

Brief but incomplete

General algorithms:

- using linear algebra (Lipshitz, 1988);
- using non-commutative Gröbner bases:
  - and elimination (Takayama, 1990);
  - and rational resolution of differential equations (Chyzak, 2000);
  - and heuristics (Koutschan, 2010).
- etc.

Algorithms for the rational case:

- univariate integrals (Bostan, Chen, Chyzak, Li, 2010);
- double integrals (Chen, Kauers, Singer, 2012).

Polynomial time computation

Main result

$F = \frac{a}{f}$ — a rational function in t and $\mathbf{x} = x_1, \ldots, x_n$

$d_{\mathbf{x}}$ — the degree of f w.r.t. $\mathbf{x}$

d_t — $\max(\deg_t f, \deg_t a)$

Hypothesis — Simplifying assumption: $\deg_{\mathbf{x}} a + n + 1 \leqslant d_{\mathbf{x}}$

Theorem (Bostan, Lairez, Salvy, 2013)

A telescoper for F can be computed using $\widetilde{\mathcal{O}}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$ operations in the base field, uniformly in all the parameters. The minimal telescoper has order $\leqslant d_{\mathbf{x}}^n$ and degree $\mathcal{O}(e^n d_{\mathbf{x}}^{3n} d_t)$.

Remark

Each side of any telescopic relation has size at least $d_{\mathbf{x}}^{(1-\varepsilon)n^2}$, generically.

Main ingredients of the algorithm

Griffiths–Dwork method for the generic case

Linear reduction used in algebraic geometry

Generalization of Hermite's reduction

Fast linear algebra on polynomial matrices

Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case

Pertubation of F with a new free variable

Homogenization

$$\tilde{F} \stackrel{\text{def}}{=} {x_0}^{-n-1} F\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = \frac{a}{f}.$$

Proposition

Homogeneous–inhomogeneous equivalence $L(t, \partial_t)$ *is a telescoper for* $\tilde{F}$ *if and only it is a telescoper for* F*.*

The degree $-n-1$ is choosen to ensure this property.

Griffiths–Dwork reduction

Input $F = a/f^\ell$ a rational function in $x_0, \ldots, x_n$

Output $[F]$ such that there exist rational functions $A_0, \ldots, A_n$ such that $F = [F] + \sum_i \partial_i A_i$

Precompute a Gröbner basis G for $(\partial_0 f, \ldots, \partial_n f)$

procedure $[\cdot](a/f^\ell)$

  if $\ell = 1$ **then return** a/f^ℓ

  Decompose a as $r + \sum_{i=0}^{n} v_i \partial_i f$ using G

  return $\frac{r}{f^\ell} + \left[\frac{1}{\ell - 1} \sum_i \frac{\partial_i v_i}{f^{\ell-1}} \right]$

Properties of the reduction

f is fixed.

- Linearity: $[\cdot]$ is linear.
- Soundness: If $[F] = 0$ then $F = \sum_i \partial_i A_i$.

(Dwork, Griffiths) Moreover, if the ideal $(\partial_0 f, \dots, \partial_n f)$ is 0-dimensional, then:

- Confinement: The image of $[\cdot]$ is finite dimensional.
- Normalization: $\left[\partial_i \left(\frac{b}{f^N}\right)\right] = 0$.

Generic case

Input $F = a/f^\ell$ a generic homogeneous rational function

Output $L(t, \partial_t)$ a telescoper for F.

procedure $\mathrm{Telesc}_{\mathrm{reg}}(F)$

  $G_0 \leftarrow [F]$

  $i \leftarrow 0$

  loop

    if $\mathrm{rank}_L(G_0, \ldots, G_i) < r + 1$ **then**

      solve $\sum_{k=0}^{r-1} a_k G_k = G_i$ w.r.t. $a_0, \ldots, a_{r-1}$ in L

      return $\partial_t^r - \sum_k a_k \partial_t^k$

    else

      $G_{r+1} \leftarrow [\partial_t G_r]$

      $r \leftarrow r + 1$

Singular case: deformation

Input $F = a/f^\ell$ a homogeneous rational function

Output $L(t, \partial_t)$ a telescoper for F.

procedure Telesc(F)

$$f_{\text{reg}} \leftarrow f + \varepsilon \sum_{i=0}^{n} x_i^{d_{\mathbf{x}}} \in K[t, \varepsilon, \mathbf{x}]$$

$$\tilde{F}_{\text{reg}} \leftarrow \frac{a}{f_{\text{reg}}^\ell}$$

return $\text{Telesc}_{\text{reg}}(F_{\text{reg}})|_{\varepsilon=0}$

The deformation method:

1. has good complexity,
2. loses minimality properties.

Timings

For a generic $\frac{a}{f^2} \in \mathbb{Q}(t, x_1, x_2)$:

$\deg_x f$	3	4	5	6
order	2	6	12	20
$\deg_t f = \deg_t a = 1$	32 *(0.4s)*	153 *(46s)*	480 *(2h)*	1175 *(150h)*
$\deg_t f = \deg_t a = 2$	66 *(0.6s)*	336 *(140s)*	1092 *(7h)*	? *()*
$\deg_t f = \deg_t a = 3$	100 *(0.9s)*	519 *(270s)*	1704 *(13h)*	? *()*

New (columns 5 and 6)

Conclusion

$$\widetilde{\mathcal{O}}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$$

- First polynomial time algorithm for rational creative telescoping
- Accurate bounds on the size of the output
- Proof that the certificate is generically way bigger that the telescoper
- On going work on the singular case