Computing with *integrals* in nonlinear algebra

#3

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Plan

#1 Differential equations as a datastructure

#2 High-precision numerical evaluation. Application in experimental maths.

#3 Formal integration: diagonals, constant terms, residues

#4 Periods of projective hypersurfaces

The period of the sine function

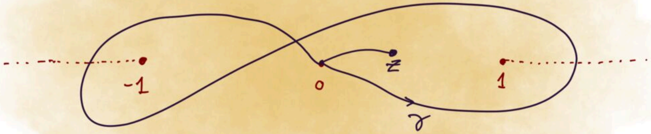
$$\sin\left(\int_0^z \frac{du}{\sqrt{1-u^2}}\right) = z \qquad \left(\text{because } \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}\right)$$

$$\sin\left(\oint_\gamma \frac{du}{\sqrt{1-u^2}} + \int_0^z \frac{du}{\sqrt{1-u^2}}\right) = z$$

So sin is $\oint_\gamma \frac{du}{\sqrt{1-u^2}}$ - periodic

$$\Vert$$

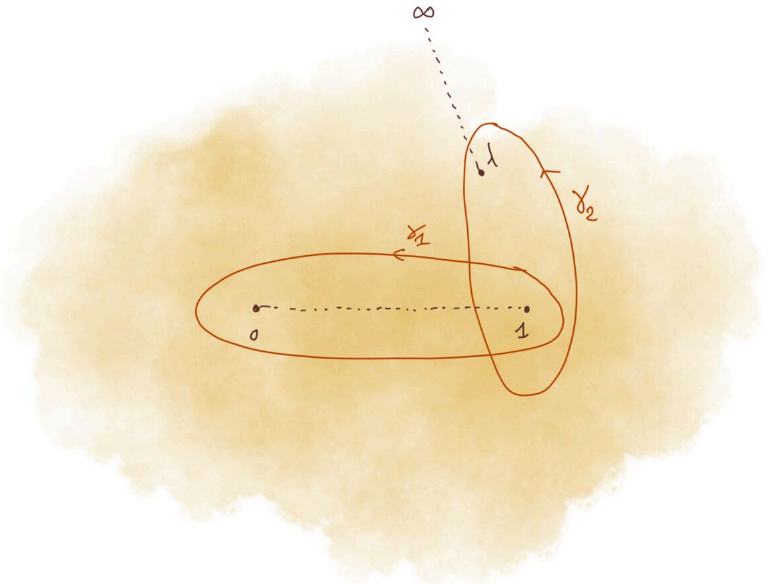
$$2\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = 2\pi$$



Elliptic periods

$$\wp_\lambda\left(\int_\infty^z \frac{du}{\sqrt{u(u-1)(u-\lambda)}}\right) = z$$

Weierstrass' elliptic function



$$\omega_i = \oint_{\gamma_i} \frac{du}{\sqrt{u(u-1)(u-\lambda)}}, \quad i = 1, 2$$

are the two periods of $\wp_\lambda$.

______ of the elliptic curve $y^2 = x(x-1)(x-\lambda)$ (E_λ)

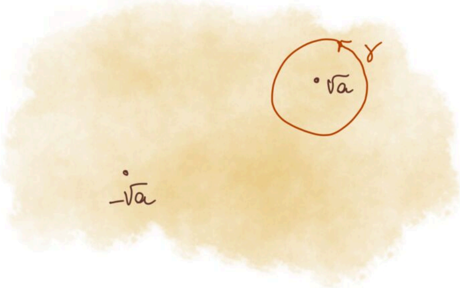
The ratio $\tau = \frac{\omega_1}{\omega_2}$ characterizes entirely the isomorphism class of E_λ.

Sagemath

From algebraic to rational functions

$$\frac{1}{2i\pi}\oint_\gamma \frac{du}{u^2-a} = \frac{1}{2\sqrt{a}}$$

$$\frac{1}{u^2-a} = \frac{1}{2\sqrt{a}}\left(\frac{1}{u-\sqrt{a}} - \frac{1}{u+\sqrt{a}}\right)$$



Therefore $$\oint \frac{dx}{\sqrt{P(x)}} = \frac{1}{i\pi}\oint\left(\oint_{\gamma_x} \frac{dy}{y^2-P(x)}\right)dx$$

$$= \frac{1}{i\pi}\oint_\Gamma \frac{dx\,dy}{y^2-P(x)}$$

2-dimensional cycle

General setting

$P \in \mathbb{C}[x_1, \ldots, x_n]$ polynomial of degree d

control the pole at ∞

We look at integrals $\int_\gamma \frac{A\, dx_1 \cdots dx_n}{P^k}$ with $A \in \mathbb{C}[x_1, \ldots, x_n]$

and $\deg A + n + 1 \leq kd$

and γ a n-cycle in $\mathbb{P}^n \setminus \overline{V(P)}$

that is a n-dimensional submanifold of $\mathbb{P}^n \setminus \overline{V(P)}$

Notations: $X = \overline{V(P)} \subseteq \mathbb{P}^n$

$U = \mathbb{P}^n \setminus X$.

$\mathbb{C}(U)_{-n-1} = \left\{ \frac{A}{P^k} \mid \deg A + n + 1 \leq kd \right\} \simeq \left\{ \frac{A_{hom}}{P_{hom}^k} \mid A_{hom} \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogeneous of degree } kd - n - 1 \right\}$

Topological reductions of periods

application of Stokes' formula

$\int_\gamma \frac{A}{P^k} d\underline{x}$ depends only on the class of γ modulo homology.

There are n-cycles $\gamma_1, \ldots, \gamma_s \subset U$ such that any other cycle η equals $a_1\gamma_1 + \cdots + a_s\gamma_s$ for some $a_i \in \mathbb{Z}$.

Analytic reduction

$$\int_\gamma \sum_i \frac{\partial F_i}{\partial x_i} d\underline{x} = 0,$$ for any $\gamma \in H_n(U)$ and any $F_0, \dots, F_n \in \mathbb{C}(U)_{-n}$

The same relation that we used for residues.

for the regular case.

Theorem (Grothendieck, Griffiths, Monsky)

the quotient space $\mathbb{C}(U)_{-n-1} \Big/ \sum_i \frac{\partial}{\partial x_i} \mathbb{C}(U)_{-n}$ is finite dimensional.

$F \mapsto F\,dx_1 \cdots dx_n$

$$= \Omega^n(U) \Big/ d\Omega^{n-1}(U) = H^n_{DeRham}(U)$$

(and this holds for any base field)

Picard-Fuchs equations

Base field $K = \mathbb{Q}(t)$

Let $F \in K(\mathcal{U}) \subseteq \mathbb{C}(t, x_1, \ldots, x_n)$

Then there is a $L \in \mathbb{Q}[t]\langle \partial_t \rangle$ s.t.

Picard-Fuchs diff. eq.

Fuchsian + rational exponents

$$L \cdot F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i \quad . \quad \text{for some } A_i \in K(\mathcal{U}).$$

This is Christol-Lipshitz theorem!

Griffiths-Dwork algorithm

INPUT : $\frac{A}{P^k} \in K(U)_{-n-1}$

ASSUMPTION : X is regular

ie $\langle \partial_0 P, \dots, \partial_n P \rangle^{rad} = \langle x_0, \dots, x_n \rangle$

OUTPUT : $\frac{B}{P^n} \in K(U)_{-n-1}$

that is a normal form modulo derivatives.

Write $A = \sum_i B_i \partial_i P + R$ using a division by a Gröbner basis of $\langle \partial_0 P, \dots, \partial_n P \rangle$

Return $\frac{R}{P^k} + \text{Griffiths-Dwork}\left(\frac{1}{k-1} \frac{\sum_i \partial_i B_i}{P^{k-1}} \right)$

$$\frac{\sum_i B_i \partial_i P}{P^k} + \sum_i \frac{\partial}{\partial x_i}\left(\frac{1}{k-1} \frac{B_i}{P^{k-1}} \right)$$

Summary

Given $P \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous,

we have a basis $\gamma_1, \dots, \gamma_s$ of $H_n(U, \mathbb{Z})$

a basis $R_1, \dots, R_r$ of $\mathbb{C}(U)_{-n-1} \big/ \sum_i \partial_i \mathbb{C}(U)_{-n} \simeq H^n_{DR}(U)$

$$\rightsquigarrow \quad \mathcal{P} = \left(\oint_{\gamma_j} R_i \, dx_1 \cdots dx_n \right)_{i,j}$$

If P depends on some parameter t, then $\mathcal{P}$ too and

$$\mathcal{P}' = A(t)\,\mathcal{P}$$

If we know the period matrix at one point, we can compute it anywhere!

The Picard group of a surface (a.k.a. the Neron-Severi group)

Let $X \subset \mathbb{P}^3$ be a smooth complex algebraic surface.

Any algebraic curve $C \subset X$ gives a 2-cycle in X

$$\mathrm{Pic}\,X = \langle\, [C] \in \underbrace{H_2(X,\mathbb{Z})}_{\simeq \mathbb{Z}^e,\text{ depends only on the degree of } X} \;|\; C \text{ algebraic curve on } X \,\rangle$$

This is not a topological invariant but an algebraic one

$\mathrm{rk}\,X$ = the rank of $\mathrm{Pic}\,X$ $\quad (\mathrm{Pic}\,X \simeq \mathbb{Z}^{\mathrm{rk}\,X})$

NB. $[X \cap \{\text{hyperplane}\}] \in \mathrm{Pic}\,X$. $\qquad \mathrm{rk}\,X \geq 1$.

A few cases

$X \subseteq \mathbb{P}^3$

$\deg X$	X	$\mathrm{Pic}\, X$	$H_2(X, \mathbb{Z})$
1	$\simeq \mathbb{P}^2$	$\mathbb{Z}$	$\mathbb{Z}$
2	$\simeq \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
3	$\simeq \mathbb{P}^2$ blown up at 6 points	$\mathbb{Z}^7$	$\mathbb{Z}^7$
4		?	$\mathbb{Z}^{22}$
5		?	$\mathbb{Z}^{53}$

Quartic surfaces

Example. (Fermat quartic)

$X = \{ w^4 + x^4 + y^4 + z^4 = 0 \}$ contains 48 lines (e.g. $x = u$, $y = v$, $z = \omega u$ and $w = \omega' v$) where $\omega^4 = \omega'^4 = -1$.

and $\mathrm{Pic}(X) = \mathbb{Z}^{20}$

But...

Theorem (Noether, Lefschetz)

If $\deg X \geq 4$, and X is very generic, then $\mathrm{Pic}\, X = \mathbb{Z}$

excludes a countable union of hypersurfaces in $\mathbb{C}[w,x,y,z]_{\deg X}$

Surprisingly difficult to find one!

$\{ wx^3 + w^3 y + y^4 + xz^3 + z^4 = 0 \}$

Smooth rational curves

Pic X comes with : - a distinguished element h, the hyperplane section.

quartic surface

- a bilinear form $\mathrm{Pic}\,X \times \mathrm{Pic}\,X \longrightarrow \mathbb{Z}$,

  the intersection form.

If $\gamma = [C] \in \mathrm{Pic}\,X$, then:

- $\gamma \cdot h = \deg C$
- $\gamma \cdot \gamma = 2p_a(C) - 2$

Classes with $\gamma \cdot \gamma = -2$ correspond to smooth rational curves!
(somewhat)

A numerical characterization of the Picard group

Lefschetz' theorem on (1,1)-form.

To simplify

Let $X = \{ P = 0 \} \subseteq \mathbb{P}^3$ be a quartic surface

$$\mathrm{Pic}(X) = \left\{ \gamma \in H_2(X, \mathbb{Z}) \;\middle|\; \int_{\mathrm{Tube}(\gamma)} \frac{dx\,dy\,dz}{P(1,x,y,z)} = 0 \right\}$$

Algorithm

1. Pick a basis $\gamma_1, \dots, \gamma_{22}$ of $H_2(X, \mathbb{Z})$
2. Compute $\displaystyle\int_{\mathrm{Tube}(\gamma_i)} \frac{dx\,dy\,dz}{P(1,x,y,z)} \in \mathbb{C}$
3. Compute $\left\{ (a_1, \dots, a_{22}) \in \mathbb{Z}^{22} \;\middle|\; \sum_i a_i \gamma_i = 0 \right\}$

Computing the lattice of integer relations

$n=2.$ Given $\alpha, \beta \in \mathbb{C}$ (say nonzero),

how to compute $\Lambda = \{ (u,v) \in \mathbb{Z}^2 \mid u\alpha + v\beta = 0 \}$?

Of course, α and β are known only approximatedly..

Two possible cases:
- $\frac{\alpha}{\beta} \notin \mathbb{Q}$ and $\operatorname{rk} \Lambda = 0$
- $\frac{\alpha}{\beta} = \frac{p}{q} \in \mathbb{Q}$ and $\Lambda = \mathbb{Z} \cdot (q, -p)$

Search for $\frac{p}{q} \in \mathbb{Q}$ s.t. $\left| \frac{\alpha}{\beta} - \frac{p}{q} \right| \ll \frac{1}{\max(|p|,|q|)^2}$ $\longrightarrow$ Found nothing

Found nothing $\nearrow$ $\frac{\alpha}{\beta} \notin \mathbb{Q}$

Found nothing $\searrow$ $\frac{\alpha}{\beta} \in \mathbb{Q}$ but precision is not enough to see it

Search $\swarrow$ There is one and $\frac{\alpha}{\beta} = \frac{p}{q}$

Search $\searrow$ There is one but $\frac{\alpha}{\beta} \neq \frac{p}{q}$

Computing the lattice of integer relations

General n. Input: $\gamma_1, \dots, \gamma_n \in \mathbb{C}$.

$$\Lambda = \left\{ \underline{a} \in \mathbb{Z}^n \;\middle|\; \sum_i a_i \gamma_i = 0 \right\}.$$

Let $M = \left\{ (\underline{a}, u, v) \in \mathbb{Z}^{n+2} \;\middle|\; \sum_i a_i \left[10^p \cdot \gamma_i \right] = u + v\sqrt{-1} \right\}$

Typically, $M = \tilde{\Lambda} \oplus G$ LLL!

$\tilde{\Lambda} \subseteq \Lambda$

sublattice generated by small vectors

Garbage
Does not contain any small vector

Triple alternative:

$\tilde{\Lambda} \simeq \Lambda$ or Λ is not gerated by small vectors or $\tilde{\Lambda}$ contains an astonishing numerical coincidence

Separation of periods

Let X be a quartic surface, and $\gamma_1, \dots, \gamma_{22}$ its periods.

Let $a_1, \dots, a_{22} \in \mathbb{Z}^{22}$ s.t. $\sum_i a_i \gamma_i$ is small.

How small does it need to be to imply that $\sum_i a_i \gamma_i = 0$?

Theorem. (Lairez, Sertöz) $\exists\, c > 0$ (depending only on X) s.t.

$$\sum_i a_i \gamma_i = 0 \quad \text{or} \quad \left| \sum_i a_i \gamma_i \right| > 2^{-c^{\max_i |a_i|^9}}$$