Finding one root of a polynomial system

How to improve the complexity?

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On a problem posed by Steve Smale

By Peter Bürgisser and Felipe Cucker

Abstract

The 17th of the problems proposed by Steve Smale for the 21st century asks for the existence of a deterministic algorithm computing an approximate solution of a system of n complex polynomials in n unknowns in time polynomial, on the average, in the size N of the input system. A partial solution to this problem was given by Carlos Beltrán and Luis Miguel Pardo who exhibited a randomized algorithm doing so. In this paper we further extend this result in several directions. Firstly, we exhibit a linear homotopy algorithm that efficiently implements a nonconstructive idea of Mike Shub. This algorithm is then used in a randomized algorithm, call it LV, à la Beltrán-Pardo. Secondly, we perform a smoothed analysis (in the sense of Spielman and Teng) of algorithm LV and prove that its smoothed complexity is polynomial in the input size and σ^{-1}, where σ controls the size of of the random perturbation of the input systems. Thirdly, we perform a condition-based analysis of LV. That is, we give a bound, for each system f, of the expected running time of LV with input f. In addition to its dependence on N this bound also depends on the condition of f. Fourthly, and to conclude, we return to Smale's 17th problem as originally formulated for deterministic algorithms. We exhibit such an algorithm and show that its average complexity is $N^{\mathcal{O}(\log\log N)}$. This is nearly a solution to Smale's 17th problem.

Contents

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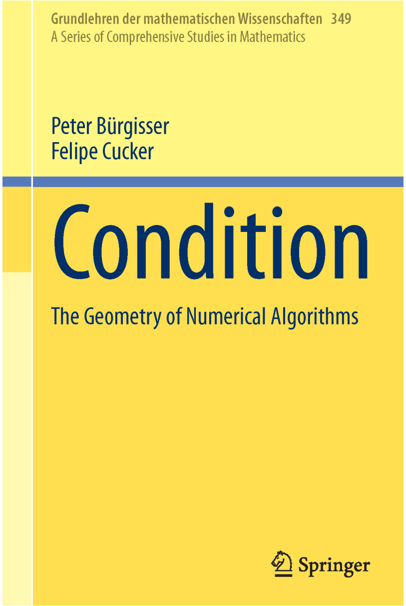
An extended abstract of this work was presented at STOC 2010 under the title "Solving Polynomial Equations in Smoothed Polynomial Time and a Near Solution to Smale's 17th Problem".



Can we compute the roots of a polynomial system in polynomial time?

Likely not, deciding *feasibility* is NP-complete.

Can we compute the complex roots of n equations in n variables in polynomial time?

No, there are too many roots.

Bézout bound vs. input size (n polynomial equations, n variables, degree δ)

degree	δ	2	n	$\delta \gg n$
input size	$n\binom{\delta+n}{n}$	$\sim \frac{1}{2} n^3$	$\sim \frac{1}{\sqrt{\pi}} n^{\frac{1}{2}} 4^n$	$\sim \frac{1}{(n-1)!} \delta^n$
#roots	δ^n	2^n	n^n	δ^n

Finding one root: a purely numerical question

#roots $\gg$ input size To compute a single root, do we have to pay for #roots?

using exact methods Having one root is having them all (generically).

using numerical methods One may approximate one root disregarding the others.

polynomial complexity? Maybe, but only with numerical methods

This is Smale's question

Now solved, let's ask for more!

Numerical continuation

F_t a polynomial system depending continuously on $t \in [0,1]$

z_0 a root of F_0

function NumericalContinuation(F_t, z_0)

$t \leftarrow 0$

$z \leftarrow z_0$

repeat

$t \leftarrow t + \Delta t$

$z \leftarrow \mathrm{Newton}(F_t, z)$

until $t \geqslant 1$

return z

end function

- Solves any generic system
- How to set the step size Δt ?
- How to choose the start system F_0?
- How to choose a path?

A short history

Average analysis

the complexity is unbounded near singular cases.

$\rightsquigarrow$ stochastic analysis

global distribution centered Gaussian in the space of all polynomial systems

local distribution non-centered Gaussian

randomized algorithms choosing the continuation path may need randomization

Lairez (2017) this can be derandomized eliminated for average analysis

$$0.50529019746531591013322667888500001621027$$

noise extraction ↑

$$x = \underline{0.604402562418089516117808124910468}\overline{6505290197465315910133226678885000016210 27}$$

truncation ↓

$$0.6044025624180895161178081249104686$$

Renegar (1987)

n complex variables

n random equations of degree δ

input size N

input distribution centered

# of steps $\mathrm{poly}(\delta^n)$, with high probability

starting system $x_1^\delta = 1, \ldots, x_n^\delta = 1$

continuation path $(1 - t)F_0 + tF_1$

previous best $\varnothing$

Shub, Smale (1994)

n complex variables

n random equations of degree δ

input size N

input distribution	centered
# of steps	$\mathrm{poly}(N)$, with high probability
starting system	not constructive
continuation path	$(1-t)F_0 + tF_1$
previous best	$\mathrm{poly}(\delta^n)$

Beltrán, Pardo (2009)

n complex variables
n random equations of degree δ
input size N

input distribution	centered
# of steps	$O(n\delta^{3/2}N)$, on average
starting system	random system, sampled directly with a root
continuation path	$(1-t)F_0 + tF_1$
previous best	$\mathrm{poly}(\delta^n) \to \mathrm{poly}(N)$

Bürgisser, Cucker (2011)

n complex variables
n random equations of degree δ
input size N

input distribution non-centered, variance σ^2, really relevant to applications!

# of steps $O(n\delta^{3/2}N/\sigma)$, on average

starting system idem Beltrán-Pardo

continuation path $(1-t)F_0 + tF_1$

previous best $\varnothing$

Armentano, Beltrán, Bürgisser, Cucker, Shub (2016)

n complex variables

n random equations of degree δ

input size N

input distribution	centered
# of steps	$O(n\delta^{3/2}N^{1/2})$, on average
starting system	idem Beltrán-Pardo
continuation path	$(1-t)F_0 + tF_1$

previous best $\mathrm{poly}(\delta^n) \to \mathrm{poly}(N) \to O(n\delta^{3/2}N)$

n complex variables

n random equations of degree δ

input size N

input distribution centered

# of steps $O(n^3\delta^2)$, on average

starting system an analogue of Beltrán-Pardo

continuation path $(f_1 \circ u_1^{1-t}, \dots, f_n \circ u_n^{1-t})$, with $u_i \in U(n+1)$
(rigid motion of each equations)

previous best $\mathrm{poly}(\delta^n) \to \mathrm{poly}(N) \to O(n\delta^{3/2}N) \to O(n\delta^{3/2}N^{1/2})$

Improving the conditioning

How to improve the complexity?

By making **bigger steps!**

$z =$ the current root

$\rho(F, z) =$ inverse of the radius of the bassin of attraction of z

$\mu(F, z) = \sup\,[\text{over } F' \sim F \text{ and } F'(z') = 0]\ \frac{\operatorname{dist}(z,z')}{\|F-F'\|}$

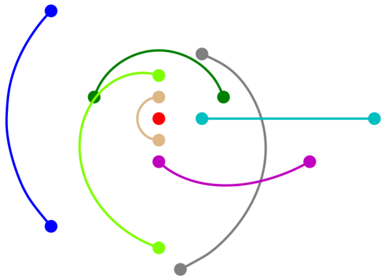
step size heuristic $$\frac{1}{\Delta t} \approx \rho(F, z) \cdot \frac{\Delta z}{\Delta t} \lessapprox \underbrace{\mu(F, z)}_{\text{loose}} \cdot \underbrace{\mu(F, z)}_{\text{sharp}}.$$

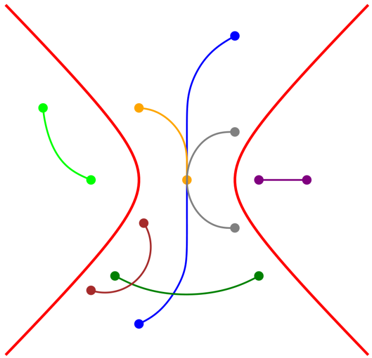
average analysis Each factor μ contributes $O(N^{1/2})$ in the average # of steps.
To go down to $\operatorname{poly}(n, \delta)$, we must improve both.

Changing the path

an old idea Can we choose a path that keeps $\mu(F, z)$ low?
i.e. that stays far from singularities?

yes! Beltrán, Shub (2009)
...but not applicable for polynomial system solving.





(Pictures by Juan Criado del Rey.)

Rigid continuation algorithm

input $f_1, \dots, f_n$, homogeneous polynomials of degree δ in $x_0, \dots, x_n$

1. Pick $x \in \mathbb{P}^n(\mathbb{C})$
2. For $1 \leqslant i \leqslant n$,
   - **a** compute one point $p_i \in \mathbb{P}^n(\mathbb{C})$ such that $f_i(p_i) = 0$
   - **b** pick $u_i \in U(n+1)$ such that $u_i(x) = p_i$.
3. Perform the numerical continuation with

$$F_t = \left(f_1 \circ u_1^{1-t}, \dots, f_n \circ u_n^{1-t}\right).$$

big win the parameter space has $O(n^3)$ dimensions,
the conditioning is $\operatorname{poly}(n)$ on average

total complexity $O(n^6 \delta^4 N) = N^{1+o(1)}$ operation on average, quasilinear

Toward structured systems

Why structured systems?

structures sparse

symmetries

low evaluation complexity black box

This includes most practical examples!

Traditional average analysis is irrelevant.

observation A $\mathrm{poly}(N)$ complexity is far from what we observe in practice.

We want $\mathrm{poly}(n, \delta)\, \mathrm{cost}$(input)

input F given as a black box function

question Can we adapt the rigid continuation algorithm?

Yes!, but with small probability of failure

difficulty Computing γ requires all coefficients, costs $N \gg \mathrm{cost}(F)$.

stochastic formulation $$\gamma(f, z) \approx \min_{\rho > 0} \frac{\mathbb{E}\left|f(z+\rho w) - f(z)\right|}{\rho^2 \|\mathrm{d}_z f\|},$$

with w uniformly distributed in the unit ball.

Stochastic optimization problem

Random black box input

input F given as a black box function, randomly distributed

question Is the average complexity $\operatorname{poly}(n, \delta) \operatorname{cost}(F)$? Watch arXiv...

random black boxes What it is?

A random model for a black box (homogeneous) polynomial:

$$f(x_0, \dots, x_n) = \operatorname{trace}\left(A_1(x_0, \dots, x_n) \cdots A_\delta(x_0, \dots, x_n)\right),$$

where the A_i are $r \times r$ matrices with degree 1 entries, coeefficients are i.i.d. Gaussian.

evaluation complexity $O(r^3\delta + r^2 n)$

The parameter r reflects the complexity of evaluating f.

Polynomially equivalent to Valiant's determinantal complexity.

Thank you!

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